TURBULENT BOUNDARY LAYER ON A FLAT PLATE IN A SUPERSONIC STREAM OF EQUILIBRIUM DISSOCIATED AND IONIZED GAS

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A method is described for calculating the turbulent boundary layer on a flat plate in an equilibrium dissociated and ionized supersonic stream. An estimate is made of the effect of ionization on the drag and heat transfer in a stream of air.

The need for earth re-entry of hypersonic vehicles has stimulated the development of methods for computing the drag and heat transfer in flow of dissociated and ionized gas mixtures. While considerable success has been achieved [1-6] in investigation of laminar flows of this type, turbulent flows have received little study. The author knows of only one paper [7] dealing with investigation of the turbulent boundary layer on a flat plate in a stream of equilibrium dissociated and ionized air. The method is predicated on the equilibrium composition, thermodynamic properties, and transfer coefficients of the mixture being known functions of temperature and pressure. But this approach cannot be used to calculate the characteristic features of turbulent heat and mass transfer in multicomponent mixtures (such as the effect of separation of elements in the laminar sublayer and the occurrence of diffusion terms in the expression for the heat flux).

The present paper suggests a method which is free of these defects.

We assume a two-layer model of the turbulent boundary layer (a laminar sublayer and a turbulent core). We put the turbulent analogs of the Prandtl and Schmidt numbers equal to 1, and we use the ambipolar approximation [1] to describe diffusion in the laminar sublayer.

Then the system of equations of the multicomponent chemically equilibrated, partially ionized, turbulent boundary layer on a flat nonconducting plate, in the absence of external electromagnetic fields and radiative energy transfer, can be obtained in a manner analogous to that used in [8] for a mixture of electrically neutral gases, and takes the following form (the average signs are omitted for brevity):

$$\partial(\rho u)/\partial x + \partial(\rho v)/\partial y = 0, \tag{1}$$

$$\rho u \frac{\partial u}{\partial x} + \rho v \frac{\partial u}{\partial y} = \frac{\partial}{\partial y}\left[(\mu + \varepsilon)\frac{\partial u}{\partial y}\right], \quad \frac{dp}{dy} = 0, \tag{2}$$

$$\rho u \frac{\partial H}{\partial x} + \rho v \frac{\partial H}{\partial y} = \frac{\partial}{\partial y}\left\{(\mu + \varepsilon)\frac{\partial H}{\partial y} + \frac{\mu}{\mathrm{Pr}}\left[\sum_{j=1}^{N}(\mathrm{Le}_j^{(a)} - 1)h_j\frac{\partial c_j}{\partial y} + (\mathrm{Pr} - 1)\frac{\partial}{\partial y}\left(\frac{u^2}{2}\right)\right]\right\}, \tag{3}$$

$$\rho u \frac{\partial c_k^*}{\partial x} + \rho v \frac{\partial c_k^*}{\partial y} = \frac{\partial}{\partial y}\left[\sum_{j=1}^{N}\left(\frac{\mu}{\mathrm{Sc}_j^{(a)}} c_{kj}\frac{\partial c_j}{\partial y}\right) + \varepsilon\frac{\partial c_k^*}{\partial y}\right], \tag{4}$$

$$(k = 1,\ 2,\ \ldots,\ N - r - 2),$$

$$K_{ps} = \prod_{j=1}^{N}(p_j)^{\nu''_{js} - \nu'_{js}} \quad (s = 1,\ \ldots,\ r), \quad \sum_{j=1}^{N} c_j = 1, \quad \sum_{j=1}^{N}\frac{c_j e_j}{m_j} = 0, \tag{5}$$

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$$\sum_{j=1}^{N} J_j = 0, \quad \sum_{j=1}^{N} \frac{J_j e_j}{m_j} = 0, \quad J_j = -\left(\frac{\mu}{\mathrm{Sc}_j^{(a)}} + \varepsilon\right)\frac{\partial c_j}{\partial y}, \tag{6}$$

$$p = \frac{\rho RT}{m}, \quad m^{-1} = \sum_{j=1}^{N} \frac{c_j}{m_j}, \quad H = \sum_{j=1}^{N} c_j h_j + \frac{u^2}{2}, \quad h_j = \int_{T_0}^{T} c_{pj}\, dT + h_j^{(0)}. \tag{7}$$

It is assumed that r independent chemical reactions occur in the boundary layer, described by the equations

$$\sum_{i=1}^{N} \nu'_{is} A_i \rightleftarrows \sum_{i=1}^{N} \nu''_{is} A_i \quad (s = 1, 2, \ldots, r). \tag{8}$$

The effective ambipolar Schmidt numbers are determined by the relations [1]

$$\mathrm{Sc}_j^{(a)} = \sum_{i=1}^{N} c_i \frac{m}{m_i} \mathrm{Sc}_{ij} + c_j \sum_{i=1}^{N} \left(A_{ij} + A_{ij}^{(e)}\right) \frac{J_i}{J_j}, \tag{9}$$

$$A_{ij} = \sum_{k=1}^{N} \left(\frac{m}{m_i} - \frac{m}{m_k}\right) c_k\, \mathrm{Sc}_{ik} - \frac{m}{m_i} \mathrm{Sc}_{ij},$$

$$A_{ij}^{(e)} = \frac{\left(e_j - \sum_{k=1}^{N} c_k e_k\right)}{m \sum_{k=1}^{N} \frac{c_k}{m_k} e_k^2} \sum_{k=1}^{N} \frac{m^2 c_k}{m_k m_i} \mathrm{Sc}_{ki} (e_k - e_i). \tag{10}$$

In the two-layer model assumed for the turbulent boundary layer, using the system of equations (1)-(7), in the laminar sublayer one must omit terms containing the turbulent viscosity ε, and in the turbulent core one must drop terms involving the molecular-transport coefficients and dimensionless groups containing them.

The boundary conditions for the system of equations (1)-(7) have the form

$$u = u_e, \quad T = T_e, \quad p = p_e, \quad c_k^* = (c_k^*)_e \ \text{for} \ y \to \infty, \tag{11}$$

$$u_{l-0} = u_{l+0}, \quad T_{l-0} = T_{l+0}, \quad (c_k^*)_{l-0} = (c_k^*)_{l+0},$$

$$(J_k^*)_{l-0} = (J_k^*)_{l+0}, \quad q_{l-0} = q_{l+0}, \quad \tau_{l-0} = \tau_{l+0} \ \text{for} \ y = y_l, \tag{12}$$

$$u = 0, \quad v = 0, \quad T = T_w, \quad J_k^* = 0 \ \text{for} \ y = 0. \tag{13}$$

In the last of conditions (13) it is postulated that the wall is impermeable and that no heterogenous chemical reactions occur at the surface.

We convert from the xy variables to the Crocco variables $\xi = x$ and $\bar{u} = u/u_e$ [9] in the system of equations (1)-(7).

It was shown in [10] that in Eqs. (3) and (4) for the energy balance and diffusion of an element, written in Crocco variables, the terms containing derivatives with respect to ξ are of order $\mathrm{Re}^{-0.75}$. Neglecting these terms for this reason and assuming that the friction stress is constant across the boundary layer [8], we obtain

$$\frac{d\bar{q}}{d\bar{u}} = 0, \quad \frac{dJ_k^*}{d\bar{u}} = 0. \tag{14}$$

Expressions for the dimensionless generalized heat flux $\bar{q}$ and the diffusion flux of the k-th element J_k^* have the form

$$\bar{q} = \frac{1}{\mathrm{Pr}}\left[\frac{d\bar{H}}{d\bar{u}} + (\mathrm{Pr}-1)\,\bar{u}\,(k-1)\,M_e^2 + \sum_{i=1}^{N} (\mathrm{Le}_j^{(a)}-1)\,h_j\,\frac{dc_j}{d\bar{u}}\right],$$

$$J_k^* = -\frac{\tau_w}{u_e}\sum_{i=1}^{N}\frac{c_{ki}}{\mathrm{Sc}_i^{(a)}}\cdot\frac{dc_i}{d\bar{u}} \quad \text{for } \bar{u} \leqslant \bar{u}_l, \tag{15}$$

$$\bar{q} = \frac{d\bar{H}}{d\bar{u}}, \quad J_k^* = -\frac{\tau_w}{u_e}\cdot\frac{dc_k^*}{d\bar{u}} \quad \text{for } \bar{u} \geqslant \bar{u}_l. \tag{16}$$

In expressions (15)-(16) $\bar{H} = H/(c_{pe}T_e)$, $\bar{h} = h/(c_{pe}T_e)$.

The boundary conditions for the system of equations (14)-(16) can easily be obtained from the corresponding conditions (11)-(13), converting to the new variables in these equations:

$$T = T_e, \quad p = p_e, \quad c_k^* = (c_k^*)_e \quad \text{for } \bar{u} = 1, \tag{17}$$

$$T_{l-0} = T_{l+0}, \ (c_k^*)_{l-0} = (c_k^*)_{l+0}, \ (J_k^*)_{l-0} = (J_k^*)_{l+0}, \ \bar{q}_{l-0} = \bar{q}_{l+0} \quad \text{for } \bar{u} = \bar{u}_l, \tag{18}$$

$$v = 0, \quad T = T_w, \quad J_k^* = 0 \quad \text{for } \bar{u} = 0. \tag{19}$$

We take the logarithmic derivative of the equilibrium conditions (5) with respect to $\bar{u}$:

$$\frac{1}{K_{ps}}\cdot\frac{dK_{ps}}{dT}\cdot\frac{dT}{d\bar{u}} = \sum_{k=1}^{N}\left\{(\vartheta_{ks}'' - \vartheta_{ks}')\left[\sum_{j=1}^{N}\frac{d}{d\bar{u}}\left(\frac{c_j}{m_j}\right)\Big/\sum_{j=1}^{N}\left(\frac{c_j}{m_j}\right) + \frac{1}{c_k}\cdot\frac{dc_k}{d\bar{u}}\right]\right\}. \tag{20}$$

In addition, we use the first integrals of the energy-balance equation and the element diffusion equation (14). Carrying out the integration and using boundary conditions (18) and (19) and expressions (15)-(16), we obtain:

in the laminar sublayer

$$\sum_{j=1}^{N}\frac{1}{\mathrm{Sc}_j^{(a)}}\,c_{kj}\,\frac{dc_j}{d\bar{u}} = 0 \quad (k = 1,\ 2,\ \ldots,\ N-r-2), \tag{21}$$

$$\frac{dT}{d\bar{u}} = \mathrm{Pr}\,\frac{c_{pe}T_e}{c_p}\left[\bar{q}_w - (k-1)\,M_e^2\bar{u} - \sum_{j=1}^{N}\frac{\bar{h}_j}{\mathrm{Sc}_j^{(a)}}\cdot\frac{dc_j}{d\bar{u}}\right] \tag{22}$$

and in the turbulent core

$$\frac{dc_k^*}{d\bar{u}} = 0, \quad \frac{dT}{d\bar{u}} = \frac{c_{pe}T_e}{c_p}\left[\bar{q}_w - (k-1)\,M_e^2\bar{u} - \sum_{j=1}^{N} h_j\,\frac{dc_j}{d\bar{u}}\right]. \tag{23}$$

To close the system of equations (20)-(23) we use the two identities of Eq. (5), first differentiated with respect to $\bar{u}$.

The system of equations obtained can be regarded as a linear algebraic system in the derivatives of the concentrations of individual components and in the temperature of the mixture with respect to velocity. This method of calculating the concentration and temperature profiles by means of the system of equations (20)-(23) is a method of successive approximations, and reduces to the following.

In the initial approximation certain arbitrary values are taken for the dimensionless heat flux to the plate surface and for the dimensionless velocities at the edge of the laminar sublayer.

The system of equations (20)-(23) is solved at the outer edge of the boundary layer (for $\bar{u} = 1$). Here the coefficients of the system are determined from the temperature and mixture composition, which are known from the boundary condition (17). The values of the derivatives $(dc_j/d\bar{u})_e$ and $(dT/d\bar{u})_e$ resulting from the solution of the system are used in a suitable numerical method of integration to determine the

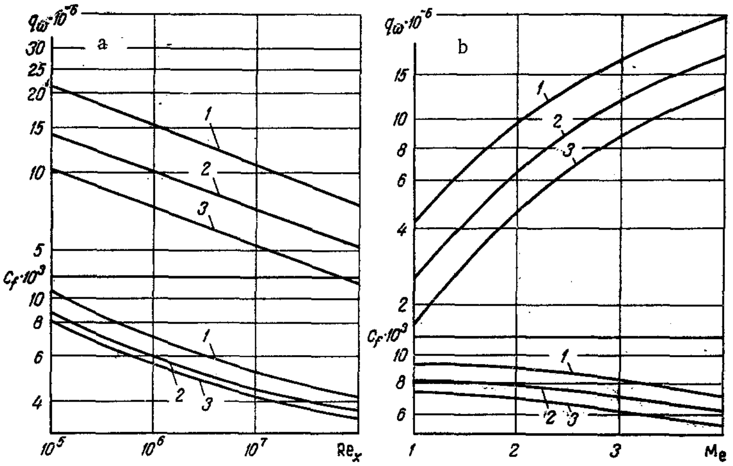

Fig. 1. Dependence of the heat flux (W/m^2) and of the friction coefficient: a) on the Reynolds number ($M_e = 3$), and b) on the Mach number ($Re_x = 4 \cdot 10^5$) for $T_w = 3000$°K, $p_e = 0.2$ bar: 1) $T_e = 14{,}000$°K; 2) 12,000; 3) 10,000.

composition and temperature at the next point of the boundary layer, $\bar{u} = 1 - \Delta\bar{u}$. For example, using the Euler method, we have

$$c_j(\bar{u}) = c_j(\bar{u} + \Delta\bar{u}) - \Delta\bar{u}\,\frac{dc_j}{d\bar{u}}(\bar{u} + \Delta\bar{u}),$$
$$T(\bar{u}) = T(\bar{u} + \Delta\bar{u}) - \Delta\bar{u}\,\frac{dT}{d\bar{u}}(\bar{u} + \Delta\bar{u}). \tag{24}$$

The above process is repeated until one passes into the laminar sublayer, i.e., for $\bar{u} \geqslant \bar{u}_l$. Thereafter the calculation procedure is altered somewhat, since the coefficients of the system of equations (20)-(23) in the laminar sublayer depend not only on the temperature and concentration of the individual mixture components, as is true in the turbulent core, but also on the derivatives $dc_j/d\bar{u}$ of the concentrations. To overcome this difficulty an iterative scheme is used. The effective ambipolar Schmidt numbers $dc_j/d\bar{u}$, which depend on $Sc_j^{(a)}$, are assumed in the initial approximation to be of arbitrary magnitude, e.g., 1. By subsequently solving the system of equations (20)-(23), we determine the values of the derivatives $dc_j/d\bar{u}$ and $dT/d\bar{u}$ at the edge of the turbulent core on the laminar sublayer side in the first approximation, and then by using Eqs. (6) and (9) we can compute the effective ambipolar Schmidt numbers in the next approximation. Then we again solve the system of equations (20)-(23), and so on, until the desired degree of accuracy is obtained.

Then Eq. (24) is used to determine the concentrations of the individual components and the mixture temperature at the next point in the laminar sublayer. This scheme of calculation is carried out right to the wall ($\bar{u} = 0$).

The above method allows us to determine the gas temperature at the wall, for a given velocity $\bar{u}_l$ at the edge of the sublayer, as a function of the dimensionless heat flux $\bar{q}_w$ to the wall.

Thus, the problem of determining $\bar{q}_w$ and the relationship between the concentration and temperature profiles and the velocity reduce to finding the roots of the nonlinear equation $f(\bar{q}_w) - T_w = 0$, which can be solved by any of the standard methods.

After calculating the concentration and temperature profiles we can use the equation of state (7) to determine the mixture density, and then use Lapin's method [8] to determine the surface-friction coefficient and the velocity at the edge of the laminar sublayer in the new approximation

$$\bar{u}_l^{(n)} = \alpha\left(\frac{2}{c_f^{(n)}} \cdot \frac{\rho_w}{\rho_e}\right)^{-\frac{1}{2}}. \tag{25}$$

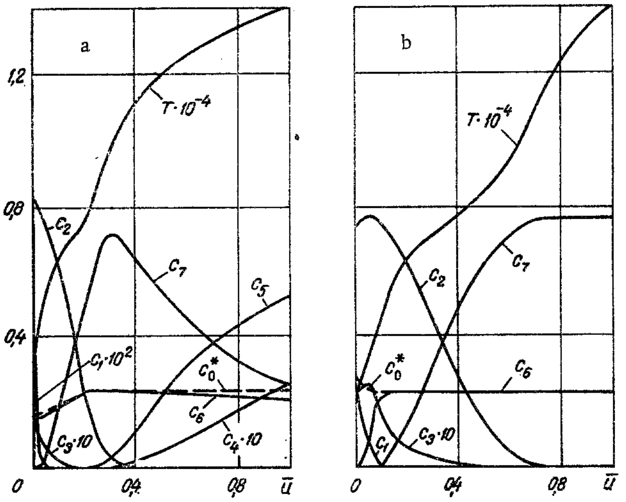

Fig. 2

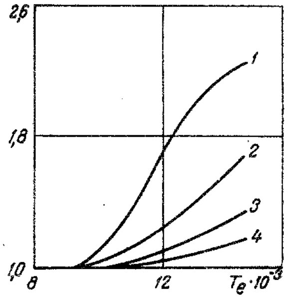

Fig. 3

Fig. 2. Typical profiles of concentrations and temperatures (°K), calculated allowing for ionization (a), and without ionization (b), for $p_e = 0.2$ bar, $M_e = 3$, $Re_x = 4 \cdot 10^5$: c_1 – O_2; c_2 – N_2, c_3 – NO; c_4 – O^+; c_5 – N^+; c_6 – O; c_7 – N.

Fig. 3. Dependence of the ratios of heat flux and friction coefficients to their corresponding values calculated without allowing for ionization, on the temperature at the outer edge of the boundary layer for $M_e = 3$, $Re_x = 4 \cdot 10^5$. $T_w = 3000$°K: 1) $q_w/(q_w)_{w.i.}$, $p_e = 0.2$ bar; 2) $q_w/(q_w)_{w.i.}$, $p_e = 1$ bar; 3) $c_f/(c_f)_{w.i.}$, $p_e = 0.2$ bar; 4) $c_f/(c_f)_{w.i.}$, $p_e = 1$ bar.

Because there is no rational theory of turbulence nor reliable test data we cannot presently estimate the effect of physical and chemical processes occurring in the flow on the structure of the turbulent boundary layer. Therefore, most of the known methods made the assumption that this effect is absent [7, 8]. Accordingly, in the specific computations we assume that the empirical constant in Eq. (25) has its value for a uniform flow, i.e., ≈11.5.

After the velocity is calculated at the edge of the laminar sublayer in the new approximation, the calculation is repeated. The criterion for ending the computation is the condition $|\bar{u}_l^n / \bar{u}_l^{(n-1)} - 1| \leqslant \varepsilon$.

This method was used to calculate the turbulent boundary layer on a flat plate in a high-temperature supersonic air stream. It was assumed that, in the range of variation of the parameters considered, the following independent thermodynamic processes occur at the outer edge of the boundary layer ($T_e = 9000$-15,000°, $p_e = 0.2$-1 bar):

$$M+O_2 \rightleftarrows M+2O,\quad M+N_2 \rightleftarrows M+2N,\quad M+NO \rightleftarrows M+N+O, \tag{26}$$

$$O+e^- \rightleftarrows O^+ + 2e^-,\quad N+e^- \rightleftarrows N^+ + 2e^-. \tag{27}$$

(Here M is a catalytic particle.)

The dependence of the constants of the equilibrium reactions (26) and (27) of the specific heats and enthalpies of the individual mixture components on the temperature were approximated using the data of [11]. The transport coefficients in the laminar sublayer were calculated using formulas of the molecular and kinetic theory of gases [12].

Here the dependence of the collision cross sections for neutral–neutral and neutral–charged particle pairs, and the cross sections for supercharging on temperature, were approximated using the data [5, 6, 13]. The cross sections for mutual collisions of charged particles were considered to be of Coulomb type.

The results of the calculations illustrating the effect of the oncoming stream parameters (Mach number $M_e = u_e/a_e$, Reynolds number $Re_x = \rho_e u_e x/\mu_e$, and temperature T_e) on the drag and the heat transfer are shown in Fig. 1 (a and b).

Figure 2 shows typical profiles of individual component concentrations and of the mixture temperature, calculated allowing for ionization and without ionization [reaction (27)]. The broken lines on Fig. 2 show the reduced curves for variation of concentration of elemental oxygen through the boundary layer. Figure 2 shows that the elemental composition of air in the laminar sublayer can vary by 20-30%; in the calculation allowing for ionization elemental nitrogen accumulates at the wall, while oxygen accumulates when ionization is not allowed for. This conclusion agrees qualitatively with the results obtained in an investigation of the laminar boundary layer in partially ionized air [3].

In order to evaluate the effect of ionization on the drag and heat transfer in the turbulent boundary layer, the friction factor c_f and the heat flux to the wall q_w were calculated, both allowing for and not allowing for ionization.

The results of the calculation are shown in Fig. 3; the ratios of q_w amd c_f to their values computed without allowing for ionization, are shown as a function of temperature, at the outer edge of the boundary layer.

The comparison was made with fixed values of Mach and Reynolds numbers. Figure 3 shows that extrapolation of the no-ionization theory to the temperature region above 10,000°K leads to an appreciable underestimate of friction and heat flux to the plate surface.

NOTATION

x, y, coordinates; u, v, velocity components; p, pressure; T, temperature; ρ, density; c_j, mass concentration of the j-th component; c_k^*, mass concentration of the k-th element; J_j, diffusion flux of the j-th component; J_k^*, diffusion flux of the k-th element; c_p, specific heat at constant pressure; m, molecular weight; e_j, charge of the j-th component; μ, dynamic viscosity; ε, turbulent viscosity; $Sc_{j,i}$, binary Schmidt number of the j-th and i-th components; c_{ki}, concentration of the k-th element in the i-th component; $h_j^{(0)}$, enthalpy of formation of the j-th component under standard conditions; K_{ps}, equilibrium constant of the s-th reaction; Pr, Prandtl number; $Le_j^{(a)}$, effective ambipolar Lewis number of the j-th component; N, number of components in the mixture; r, number of chemical reactions occurring in the boundary layer. Indices: i, j, components; k, elements; e, $l-0$, $l+0$, w, parameters at the outer edge of the boundary layer, at the edge of the laminar sublayer (approaching it from the wall side and from the outer side), and at the wall, respectively; the symbols without subscripts refer to the mixture as a whole.

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